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Polynomial Expansions of Solutions of Cauchy Problems That Involve One Space Variable

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Polynomial sets and polynomial expansion theorems are developed for solutions of Cauchy problems of the form $\partial^n u(x,t)/\partial t^n = \pm D^m u(x,t)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = \phi_j(x)$, $j = 0, 1, \ldots, n-1$. If $m > n$, then the data functions must be entire of growth $\rho = s/(s-1)$ where $s = m/n$ if n divides m and of growth $\rho = (s+1)/s$ if $m = ns + r$ where $1 \le r \le n-1$. If $n \ge m$, then R-analyticity of the $\phi_j(x)$ will suffice. © 1997 Academic Press

1. INTRODUCTION

Let m and n be a pair of positive integers and let $D = \partial/\partial x$. We will be concerned with developing polynomial expansion theorems for solutions of a class of Cauchy problems having the forms

$$\partial^n u(x,t)/\partial t^n = \pm D^m u(x,t), \qquad \partial^j u(x,t)/\partial t^j\,|_{t=0} = \phi_j(x),$$
$$j = 0, 1, \ldots, n-1, \quad (1.1)$$

where $\phi_j(x) = \sum_{k=0}^{\infty} a_{jk} x^k$. Associated with these problems are natural sets of polynomials in x and t, related to hypergeometric type polynomials, which can be constructed by means of generating functions. Depending upon the relative sizes of m and n in (1.1), the bounds on these polynomials may permit the $\phi_j(x)$ to be R-analytic in a region or else force them to be entire of appropriate growth. The simple quasi inner product will be used to obtain integral forms for these polynomials. From these, one can

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then deduce bounds on the polynomials which permit establishing the expansion theorems [4, 5]. The method used can also be applied to certain higher order Euler Poisson Darboux type problems as well as to the problems obtained by replacing $\pm D^m$ by $\pm t^p D^m$ in the equation in (1.1) where p is a non-negative integer.

The development of expansion theorems for solutions of special cases of (1.1) has an extensive history of which we review only a small part. The choices $n = 1$, $m = 2$ in (1.1) with the plus sign yield the classical heat problem which was treated in detail by P. C. Rosenbloom and D. V. Widder [14]. They proved that expansions in terms of heat polynomials require that the underlying initial data be entire of growth (ρ, τ) with $\rho \leq 2$. When $\rho = 2$, the region of convergence reduces to a time strip whose width is determined by the type τ of this entire data function. They also introduced the Appell transforms of these polynomials, called associated functions, to obtain expansions of heat functions in halfplanes $t > h$ where h is determined by the growth and type of a related entire function. These associated functions were also expressed as derivatives of the fundamental solution of the heat equation. The heat polynomials and associated functions were expressed in terms of the Hermite polynomials and the bounds and asymptotics for these Hermite polynomials [12, 15] played a central role in proving the expansion theorems. In [11], A. Hopper treated the class of problems (1.1) in which the underlying equation is given by $u_t(x,t) = (-1)^{j+1} D^{2j} u(x,t)$, $j = 1, 2, \ldots,$ by calling upon the results of [13] on the fundamental solutions for these problems. The polynomials used were generalized Hermite polynomials (see [2, 8]) and associated functions were obtained by computing appropriate derivatives of the fundamental solutions. Somewhat related studies have been carried out by D. Haimo and C. Markett [10]. Also, see [1, 9] for similar treatments of expansion theorems for solutions of the radial heat equation $u_t(r,t) = D_r^2 u(r,t) + (\mu - 1) r^{-1} D_r u(r,t)$ in terms of radial heat polynomials and associated functions. These special solution sets were expressed in terms of the generalized Laguerre polynomials and the bounds on them were vital in establishing the expansion results. In the cases of the wave and Laplace problems $u_{tt}(x,t) = \pm D^2 u$, $u(x,0) = \phi(x)$, $u_t(x,0) = 0$, Widder employed the wave polynomials $w_n(x,t) = [(x+t)^n + (x-t)^n]/2$ and the Laplace polynomials $l_n(x,t) = [(x+it)^n + (x-it)^n]/2$. If $\phi(x)$ is analytic in a region centered at $x = 0$ with radius of convergence R, Widder showed that a representation of the solution of the wave problem in terms of the $w_n(x,t)$ converges in the open square $|x| + |t| < R$ and that a representation of the solution of the Laplace problem in terms of the $l_n(x,t)$ converges in the open disk $x^2 + t^2 < R^2$. For the case $u(x,0) = 0$ and $u_t(x,0) = \phi(x)$, similar results were obtained. Finally, in [6], polynomial solution sets for a variety of hyperbolic, elliptic, and mixed problems were constructed in terms of the Jacobi polynomials. The asymptotic bounds on these Jacobi polynomials [15] permitted deducing convergence regions for expansions of solutions corresponding to analytic data in terms of these polynomial solutions. The expansion results obtained for solutions of these various partial differential equations then permitted the development of analogous function theories for these equations [7].

The vast majority of problems of type (1.1) fail to have fundamental solutions. Moreover, there are generally no transformations that map one solution of the underlying equation into another solution (such as the Appell and Kelvin transforms for the heat and Laplace equations). Finally, the polynomial solution sets associated with these problems are not usually related to the classical orthogonal polynomials for which bounds and asymptotic estimates are available. Thus, for the general Cauchy problem (1.1), we cannot introduce "associated functions" as was done for the standard heat problem. We can, however, develop a bound on a polynomial solution by writing that solution as a multiple complex integral, through the use of the quasi inner product, and then bounding that integral. This approach was used in [4] to give new developments of polynomial expansion theorems for solutions of second order initial value problems that involved the wave, the Laplace, and the Euler−Poisson−Darboux equations. This approach also permitted treating the Yukawa problem in which the solution functions corresponding to polynomial data were not polynomials. The factor switching property of the simple quasi inner product played a central role in writing suitable forms for these complex integrals. This property and its many uses were considered in some detail in [5]. As we shall see in the case $n = 1$, the use of the switching property permits us to determine the largest possible growth of entire functions that can be used as data to define a solution of (1.1).

In Section 2, we briefly recall the definition of (i) the growth and type of an entire function and (ii) the quasi inner product (qip) and its essential properties including the switching property. In addition, we note a number of qip reduction relations for formal solution operators for the problems (1.1) as given in [5]. Further, we recall a vector notation introduced in [5] for integration variables and include some special formulas for these solution operators when $n = 1$. Generating functions for polynomial solutions $H_j^{m,\pm}(x,t)$ of (1.1) when $n = 1$ are developed in Section 3 and from these the $H_j^{m,\pm}(x,t)$ are obtained explicitly. Two basic inequalities are employed to show that if $\phi(x) = \sum_{j=0}^{\infty} a_j x^j$ is entire of growth (ρ, τ) with $\rho < m/(m-1)$, then the series $\sum_{j=0}^{\infty} a_j H_j^{m,\pm}(x,t)$ defines a solution of (1.1) for all t when $n = 1$. When $\rho = m/(m-1)$, this series converges in a time strip determined by τ (the inequalities employed do not necessarily yield the optimal width for this strip when $m \geq 3$). In a brief Section 4, we write out the generating functions for the special polynomial solution sets

$P_k^{m,n,j,\pm}(x,t)$ corresponding to the cases $n \geq 2$ in (1.1) and then give explicit formulas for these polynomials. When $n \geq m$, the reduction formulas will show (in Section 5) that it suffices to select the data functions in (1.1) to be analytic in a region that includes the origin. Then, a variety of regions of convergence in the (x,t) for the expansions will be determined depending upon any restrictions imposed on the variable t. Finally, in Section 6 we discuss the case of (1.1) when $n < m$. The data functions for these types of problems must be entire. The reduction relations of Sections 2 and 3 permit determining the optimal growth rates for these functions.

2. SOME PRELIMINARIES

In this section, we recall some of the analytical notions that will play vital roles in developing expansion theorems, determining regions of convergence of expansions, and obtaining appropriate growth for entire functions that are permitted to be data functions for a variety of the problems (1.1).

A. *Analytic and Entire Functions*. For the later purpose of developing expansion theorems, we say that a function $f(x)$ is R-analytic for $|x| < R$ if the extended function $f(z)$ in the complex plane is analytic in a disk centered at $z = 0$ and of radius R. Similarly, we say that the function $f(x) = \sum_{j=0}^{\infty} a_j x^j$ is entire if $f(z)$ is analytic for all z. We also say that $f(x)$ is entire of growth (ρ, τ) with $\rho > 0$ and $\tau \geq 0$ if

$$\limsup_{j \to \infty} (j/e\rho) \cdot |a_j|^{\rho/j} = \tau \tag{2.1}$$

(we usually refer to ρ as the growth and τ as the type). This implies the existence of a positive constant M such that $|f(z)| \leq Me^{\tau|z|^{\rho}}$ for all complex z and hence all real x.

B. *Quasi Inner Products*. Let $f_j(z_j)$, $j = 1, 2$, denote a pair of functions in the complex variables z_j that are analytic in disks D_j centered at their respective origins and let $f_j(z_j) = \sum_{k=0}^{\infty} a_k^j z_j^k$. We define the *quasi inner product* of these f_j by the relation

$$\begin{aligned} f_1(\alpha_1 z_1) \circ f_2(\alpha_2 z_2) &= \frac{1}{2\pi} \int_0^{2\pi} f_1\left(\alpha_1 z_1 e^{i\theta}\right) f_2\left(\alpha_2 z_2 e^{-i\theta}\right) d\theta \\ &= \sum_{k=0}^{\infty} a_k^1 a_k^2 \alpha_1^k \alpha_2^k z_1^k z_2^k, \end{aligned} \tag{2.2}$$

where the α_j's are complex scalars and $\alpha_j z_j \in D_j$, $j = 1, 2$. From this it follows that if z_j and $Z_j \in D_j$ for $j = 1, 2$ and if $z_1 z_2 = Z_1 Z_2$, then $f_1(Z_1) \circ f_2(Z_2) = f_1(z_1) \circ f_2(z_2)$. If both of the $f_j(z_j)$ are entire in their respective variables, this last relation holds without restriction and we can write

$$f_1(\alpha_1 \alpha_2 z_1) \circ f_2(z_2) = f_1(\alpha_1 z_1) \circ f_2(\alpha_2 z_2). \tag{2.3}$$

This is called *the factor switching property for qip's*. For the problems (1.1), the formal series solution operators can be expressed in terms of qip's of entire functions as in (2.3) with one or both of the scalars α_j replaced by linear differential operators. That relation permits moving factors of a differential operator from the argument of one operator to the argument of the other operator. This was used repeatedly in [4, 5] to solve problems of type (1.1). Here, we review notions associated with these hypergeometric type solution operators and their reduction to exponential (and, hence, complex translation) operators.

C. *Formal Solution Operators*. In [5], we observed that the various formal series solution operators of "hypergeometric" type for problems such as (1.1) can be shown to have the forms

$$O(\pm t^* D^m; \nu_1, \nu_2, \dots, \nu_n) = \sum_{k=0}^{\infty} \frac{(\pm t^* D^m)^k}{(\nu_1)_k \cdots (\nu_n)_k}, \tag{2.4}$$

where t^* is a simple function of t and the ν_j's are real constants with $\nu_j \geq 1$. By application of the qip and a repeated use of the property (2.3), we showed how to reduce the calculation of $O(\phi(x))$ to the evaluation of multiple integrals of complex translations of $\phi(x)$. In the following, we review the evaluation of component operators of O and then write a qip reduction formula for that operator. These results will be applied regularly in Sections 3, 5, and 6 to deduce bounds on polynomial solutions of the problems (1.1).

(i) *Exponential Operators*. Let $\phi(x)$ be analytic in x for $|x| < R$ and let α be a complex scalar. We define the simple exponential operator $e^{\alpha D}$ by the usual relation $e^{\alpha D}\phi(x) = \phi(x + \alpha)$ which is valid if $|x + \alpha| < R$. If $\phi(x)$ is entire, this translation holds for all x. Next, let $\phi(x)$ be entire of appropriate growth. We consider evaluating $e^{\alpha D^l}\phi(x)$ where l is a positive integer ≥ 2. We showed in [5], by using (2.3), that

$$e^{\alpha D^l}\phi(x) = \int_0^{\infty} e^{-\zeta_1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{\alpha^{1/l} \zeta_1^{(l-1)/l} e^{i\theta_1} D} e^{\alpha^{(l-1)/l} \zeta_1^{1/l} e^{-i\theta_1} D^{l-1}} \phi(x)\, d\theta_1 \right\} d\zeta_1. \tag{2.5}$$

If $l = 2$, the exponential operator in the inner integral reduces to the translation operator $e^{2\alpha^{1/2}\zeta_1^{1/2}\cos(\theta_1)D}$ and the reduction is complete. If $l > 2$, we repeat applying the reduction formula (2.5) to the function $e^{\alpha^{(l-1)/l}\zeta_1^{1/l}e^{-i\theta_1}D^{l-1}}\phi(x)$. After $l-2$ further reductions of this type, we finally write

$$e^{\alpha D^l}\phi(x) = \frac{1}{(2\pi)^{l-1}}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\sum_{j=1}^{l-1}\zeta_j}\Phi(x, Z_{l-1}, \Theta_{l-1})\, d\Theta_{l-1}\, dZ_{l-1}, \tag{2.6}$$

where the function $\Phi(x, Z_{l-1}, \Theta_{l-1})$ is given by

$$\phi\Big(x + \alpha^{1/l}\Big(\zeta_1^{(l-1)/l}e^{i\theta_1} + \zeta_1^{1/l}\zeta_2^{(l-2)/l}e^{i(\theta_2-\theta_1)} + \zeta_2^{2/l}\zeta_3^{(l-3)/l}e^{i(\theta_3-\theta_2)} + \cdots + \zeta_{l-1}^{(l-1)/l}e^{-i\theta_{l-1}}\Big)\Big).$$

In (2.6), Θ_{l-1} denotes the vector $(\theta_1, \theta_2, \ldots, \theta_{l-1})$, Z_{l-1} denotes the vector $(\zeta_1, \zeta_2, \ldots, \zeta_{l-1})$, $d\Theta_{l-1} = d\theta_1 d\theta_2 \cdots d\theta_{l-1}$, and $dZ_{l-1} = d\zeta_1 d\zeta_2 \cdots d\zeta_{l-1}$. Finally $\int_{[0,2\pi]} F(\Theta_{l-1})\, d\Theta_{l-1}$ denotes the $(l-1)$-fold integral $\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} F(\Theta_{l-1})\, d\theta_1\, d\theta_2 \cdots d\theta_{l-1}$ and $\int_{[0,\infty)} G(Z_{l-1})\, dZ_{l-1}$ denotes the $(l-1)$-fold integral $\int_0^\infty\int_0^\infty\cdots\int_0^\infty G(Z_{l-1})\, d\zeta_1\, d\zeta_2 \cdots d\zeta_{l-1}$. The integral in the right member of (2.6) is defined for all complex α if $\phi(x)$ is entire of growth (ρ, τ) with $0 < \rho < l/(l-1)$ and for restricted values of α if $\rho = l/(l-1)$ (see Section 3 for some calculations pertinent to this).

(ii) *The g_a Operator.* Let $a > 1$ and let $g_a(x) = \sum_{k=0}^\infty x^k/(a)_k$. Then it is not difficult to show that

$$g_a(x) = (a-1)\int_0^1 (1-\sigma)^{(a-2)}e^{x\sigma}\, d\sigma. \tag{2.7}$$

From this, we see that we can reduce the evaluation of $g_a(\alpha D^l)\phi(x)$ to the evaluation of $(a-1)\int_0^1 (1-\sigma)^{(a-2)}e^{\alpha\sigma D^l}\phi(x)\, d\sigma$ which can now be carried out using the reductions given for exponential operators in (i). Note that $\phi(x)$ must be entire with $0 < \rho \le l/(l-1)$.

(iii) *Reductions of the O Operator.* For the moment, let S denote a complex number and suppose that it can be factored as $S = S_1 S_2$. It is easy to show that

$$\begin{aligned} O(S; \nu_1, \nu_2, \ldots, \nu_n) &= g_{\nu_1}(S_1) \circ O(S_2; \nu_2, \ldots, \nu_n) \\ &= \frac{1}{2\pi}\int_0^{2\pi} g_{\nu_1}\big(S_1 e^{i\theta_1}\big) O\big(S_2 e^{-i\theta_1}; \nu_2, \ldots, \nu_n\big)\, d\theta_1. \end{aligned} \tag{2.8}$$

If S_2 can be written as $S_3 S_4$, then we can re-apply (2.8) to the function O appearing in the last member of (2.8). A repetition of this permits reducing $O(S; \nu_1, \nu_2, \dots, \nu_n)$ to a multiple integral in which the integrand involves a product of the n functions g_{ν_j}, $j = 1, 2, \dots, n$. In view of the fact that the $g_{\nu_j}(x)$ are expressible as integrals of the exponential functions, it follows that $O(S; \nu_1, \nu_2, \dots, \nu_n)$ can be expressed as a multiple integral of a product of exponential functions involving integrations on $(n-1)$ variables θ_j and, at most, n integrations on variables σ_j over intervals $[0,1]$ as in (2.6). We can use (2.8) repeatedly to obtain multiple integrals for evaluating $O(\alpha D^m; \nu_1, \nu_2, \dots, \nu_n)\phi(x)$. The reader is referred to [5] for examples of these types of reductions.

3. GENERALIZED HEAT PROBLEMS

We now consider the generalized heat problem

$$u_t(x,t) = \pm D^m u(x,t), u(x,0) = \phi(x) \qquad \text{with } m \geq 2 \tag{3.1}$$

in which $\phi(x)$ is an entire function whose optimal growth is to be determined. The solution of this can be expressed formally as

$$u(x,t) = e^{\pm tD^m} \phi(x). \tag{3.2}$$

Prior to developing polynomial expansion theorems for solutions of (3.1), we must construct the polynomial solution sets and determine bounds on its members by application of some basic inequalities.

A. *Generalized Heat Polynomials*. We let $H_k^{m,\pm}(x,t)$ denote a solution of (3.1) corresponding the choice $\phi(x) = x^k$ in (3.1), i.e., $H_k^{m,\pm}(x,t) = e^{\pm tD^m} \cdot x^k$. To determine explicit forms for these $H_k^{m,\pm}(x,t)$, we make use of the generating function, $G(x,t,a) = e^{ax \pm ta^m}$ which corresponds to the choice $\phi(x) = e^{ax}$. Formally applying the operator $e^{\pm tD^m}$ termwise to the series $\sum_{k=0}^{\infty} a^k x^k / k!$ yields $e^{\pm tD^m} e^{ax} = \sum_{k=0}^{\infty} (e^{\pm tD_x} x^k) a^k / k!$ and, hence, the relation

$$G(x,t,a) = \sum_{k=0}^{\infty} H_k^{m,\pm}(x,t) a^k / k!. \tag{3.3}$$

But we also have

$$e^{ax}e^{\pm ta^m} = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{x^j(\pm t)^l a^{ml+j}}{j!\cdot l!} = \sum_{k=0}^{\infty}\frac{a^k}{k!}\left(\sum_{j=0}^{[k/m]}\frac{k!\cdot x^{k-mj}(\pm t)^j}{j!\cdot(k-mj)!}\right), \tag{3.4}$$

where the last term follows by making the change of summation variables $ml + j = k$ in the second member of (3.4). Comparing (3.3) and (3.4), we find

$$H_k^{m,\pm}(x,t) = \sum_{j=0}^{[k/m]}\frac{k!\cdot(\pm t)^j x^{k-mj}}{j!\cdot(k-mj)!}. \tag{3.5}$$

When $m = 2$ and the plus sign is selected, the polynomials reduce to the heat polynomials $\nu_k(x,t)$ employed in [14], i.e., $H_k^{2,+}(x,t) = \nu_k(x,t)$. When $m \geq 3$, these generalized heat polynomials are related to the polynomials $g_k^m(y)$ studied by H. Gould and A. Hopper [8]. These are generalizations of the Hermite polynomials which are defined by the generating function $G_1(y, A) = e^{mAy - A^m} = \sum_{k=0}^{\infty} g_k^m(y)A^k/k!$. By rewriting $G(x, t, a)$, it is not difficult to show that $H_k^{m,\pm}(x, t) = (\mp t)^{k/m} g_k^m(x/[m(\mp t)^{1/m}])$ [2]. This relation was employed in [11] for problems (3.1) that have fundamental solutions.

B. *Some Inequalities*. We now recall a pair of basic inequalities needed to estimate bounds on the polynomials $H_k^{m,\pm}(x,t)$ as well as on other polynomials to be considered later. These are given by

$$\begin{aligned} &\text{(a)} \quad (|x| + |y|)^{\rho} \leq 2^{\rho-1}(|x|^{\rho} + |y|^{\rho}) && \text{if } 1 \leq \rho \leq 2 \\ &\text{(b)} \quad |z|^n \leq \left(\frac{n}{\rho T}\right)^{n/\rho} e^{-n/\rho} e^{T|z|^{\rho}} && \text{for } \rho > 0, T > 0. \end{aligned} \tag{3.6}$$

The proof of (a) is straightforward and the proof of (b) is given in [3].

C. *Generalized Heat Polynomial Bounds*. We now make use of (2.5) and the switching property for qip's to write integral formulas for the $H_k^{m,\pm}(x,t)$ when $t \geq 0$ and then use the inequalities (3.6) to obtain bounds for the $H_k^{m,\pm}(x,t)$ when $m \geq 2$. The case $t < 0$ can be handled similarly by replacing $\pm t$ by $\mp(-t)$ in the integrals below. The bounds for this case are the same as those obtained below. Using (3.2), we get

$$\begin{aligned} H_k^{m,\pm}(x,t) &= e^{\pm tD^m}x^k = \frac{1}{2\pi}\int_0^\infty e^{-\zeta_1}\left\{\int_0^{2\pi} e^{\pm te^{i\theta_1}D^m} e^{\zeta_1 e^{-i\theta_1}} x^k \, d\theta_1\right\} d\zeta_1 \\ &= \frac{1}{2\pi}\int_0^\infty e^{-\zeta_1}\left\{\int_0^{2\pi} e^{\pm t^{1/m}\zeta_1^{(m-1)/m} e^{i\theta_1} D} e^{t^{(m-1)/m}\zeta_1^{1/m} e^{-i\theta_1} D^{m-1}} x^k \, d\theta_1\right\} d\zeta_1. \end{aligned} \tag{3.7}$$

Repeating the same type of reduction on the second exponential operator in this last integral as was done to deduce (2.6) from (2.5), we can finally show that

$$H_k^{m,\pm}(x,t) = \frac{1}{(2\pi)^{m-1}}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\sum_{j=1}^{m-1}\zeta_j}\,\Omega(x,t,Z_{m-1},\Theta_{m-1})\,d\Theta_{m-1}\,dZ_{m-1}, \tag{3.8}$$

where

$$\begin{aligned} \Omega(x,t,Z_{m-1},\Theta_{m-1}) &= \Big[x + t^{1/m}\Big(\pm\zeta_1^{(m-1)/m}e^{i\theta_1} + \zeta_1^{1/m}\zeta_2^{(m-2)/m}e^{i(\theta_2-\theta_1)} \\ &\qquad + \cdots + \zeta_{m-1}^{(m-1)/m}e^{-i\theta_{m-1}}\Big)\Big]^k. \end{aligned} \tag{3.9}$$

From (3.8), we find that

$$\begin{aligned} \left|H_k^{m,\pm}(x,t)\right| &\le \frac{1}{(2\pi)^{m-1}}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\sum_{j=1}^{m-1}\zeta_j}\left|\Omega(x,t,Z_{m-1},\Theta_{m-1})\right| d\Theta_{m-1}\,dZ_{m-1} \\ &\le \int_{[0,\infty)} e^{-\sum_{j=1}^{m-1}\zeta_j}\Big(|x| + t^{1/m}\Big(\zeta_1^{(m-1)/m} + \zeta_1^{1/m}\zeta_2^{(m-2)/m} \\ &\qquad + \cdots + \zeta_{m-1}^{(m-1)/m}\Big)\Big)^k\,dZ_{m-1} \\ &\le \int_{[0,\infty)} e^{-\sum_{j=1}^{m-1}\zeta_j}\Big(|x| + mt^{1/m}\big(\zeta_1+\zeta_2+\cdots+\zeta_{m-1}\big)^{(m-1)/m}\Big)^\rho\,dZ_{m-1}. \end{aligned}$$

(The last inequality in this follows from the previous one by observing that $\zeta_1^{(m-1/m} \le (\zeta_1 + \zeta_2 + \cdots + \zeta_{m-1})^{(m-1)/m}$, $\zeta_1^{1/m}\zeta_2^{(m-2)/m} \le (\zeta_1 + \cdots + \zeta_{m-1})^{1/m}(\zeta_1 + \cdots + \zeta_{m-1})^{(m-2)/m} = (\zeta_1 + \zeta_2 + \cdots + \zeta_{m-1})^{(m-1)/m}$, etc.) We now apply the inequality (3.6)(b) with $\rho \ge 1$ to the non-exponential term in the last integral of this inequality to obtain

$$\begin{aligned}
&\left|H_k^{m,\pm}(x,t)\right| \\
&\quad\le \left(\frac{k}{\rho T}\right)^{k/\rho} e^{-k/\rho}\int_{[0,\infty)} e^{-\sum_{j=1}^{m-1}\zeta_j} e^{T[|x|+mt^{1/m}(\zeta_1+\zeta_2+\cdots+\zeta_{m-1})^{(m-1)/m}]^\rho}\, dZ_{p-1} \\
&\quad\le \left(\frac{k}{\rho T}\right)^{k/\rho} e^{-k/\rho} e^{2^{\rho-1}T|x|^\rho} f(t)
\end{aligned} \tag{3.10}$$

with $f(t) = \int_{[0,\infty)} e^{-\sum_{j=1}^{m-1}\zeta_j} e^{2^{\rho-1}Tm^\rho t^{\rho/m}(\zeta_1+\zeta_2+\cdots+\zeta_{m-1})^{\rho(m-1)/m}}\, dZ_{m-1}$ (note that this is independent of k). The last member of (3.10) follows from the second member by applying (3.6)(a). We observe that the improper integral defining $f(t)$ converges for all t if $\rho(m-1)/m < 1$ or if $\rho < m/(m-1)$. On the other hand, if $\rho = m/(m-1)$, it is easy to show that this integral converges provided that $t < (2T^{m-1}m^m)^{-1}$. We will make use of the bound (3.10) for proving an expansion theorem below. The choice of T in this is related to the type τ of the entire function appearing in the data.

D. *Generalized Heat Expansions*. We now consider the problem (3.1) where $\phi(x)$ is entire with $\phi(x) = \sum_{k=0}^\infty a_k x^k$ and $\limsup_{n\to\infty}(k/e\rho)|a_k|^{\rho/k} = \tau$. Given $\varepsilon > 0$, there exists K such that if $k \ge K$, then $|a_k| \le (e\rho/k)^{k/\rho}(\tau+\varepsilon)^{k/\rho}$ where $0 < \rho \le m/(m-1)$. If $k \ge K$, it follows from (3.10) that

$$|a_k|\cdot\left|H_k^{m,\pm}(x,t)\right| \le e^{2^{\rho-1}T|x|^\rho} f(t)\left((\tau+\varepsilon)/T\right)^{k/\rho}.$$

Select $T > \tau + \varepsilon$ and first assume that $\rho < m/(m-1)$. Let N_1 and N_2 be a pair of large positive numbers and make the restrictions $|x| \le N_1$ and $|t| \le N_2$. Then it follows that the series $\sum_{k=K}^\infty |a_k|\cdot|H_k^{m,\pm}(x,t)|$ converges uniformly for (x,t) in the closed rectangle $R_1 = [-N_1, N_1]\times[-N_2, N_2]$. Similarly, if $\rho = m/(m-1)$, we can show that this series converges uniformly in the closed rectangle $R_2 = [-N_1, N_1]\times[-N_3, N_3]$ where N_3 is an arbitrary positive number satisfying $N_3 < (2T^{m-1}m^m)^{-1}$. This suggests the following:

THEOREM 3.1. *Let $\phi(x)$ be entire of growth (ρ,τ) with $\rho \le m/(m-1)$ where $m \ge 2$. Then the series $\sum_{k=0}^\infty a_k H_k^{m,\pm}(x,t)$ converges to a solution of the Cauchy problem $u_t(x,t) = \pm D^m u(x,t)$, $u(x,0) = \phi(x)$ for all t if $\rho < m/(m-1)$. If $\rho = m/(m-1)$, then this series converges to a solution of this Cauchy problem for $|t| < (2\tau^{m-1}m^m)^{-1}$.*

Proof. In view of the above discussion, we must establish that the derived series also converge uniformly. From the generating function $e^{ax \pm a^m t}$, it is not difficult to show that

$$\begin{aligned} &\text{(a)} \quad DH_k^{m,\pm}(x,t) = kH_{k-1}^{m,\pm}(x,t) \\ &\text{(b)} \quad \partial H_k^{m,\pm}(x,t)/\partial t = \pm k(k-1)\cdots(k-m+1)H_{k-m}^{m,\pm}(x,t). \end{aligned} \tag{3.11}$$

From the first of these and the bounds on $H_k^{m,\pm}(x,t)$, it can be shown that

$$|a_k| \cdot \left| DH_k^{m,\pm}(x,t) \right| \leq e^{2^{\rho-1}T|x|^\rho} f(t) k((\tau+\varepsilon)/T)^{(k-1)/\rho}.$$

By the ratio test, the constant term series $\sum_{k=K}^{\infty} k((\tau+\varepsilon)/T)^{(k-1)/\rho}$ converges if $\tau + \varepsilon < T$. Hence, the series $\sum_{k=0}^{\infty} a_k DH_k^{m,\pm}(x,t)$ converges uniformly in R_1 if $\rho < m/(m-1)$ and uniformly in R_2 if $\rho = m/(m-1)$ if we use the above restriction on T. In a similar way, one can also prove that the two series $\sum_{k=0}^{\infty} a_k D^2 H_k^{m,\pm}(x,t)$ and $\sum_{k=0}^{\infty} a_k(\partial H_k^{m,\pm}(x,t)/\partial t)$ also converge uniformly in the region R_1 or the region R_2 depending upon the value of ρ. Then the $H_k^{m,\pm}(x,t)$ satisfy the equation given in the statement of the theorem and $\sum_{k=0}^{\infty} a_k H_k^{m,\pm}(x,0) = \sum_{k=0}^{\infty} a_k x^k = \phi(x)$ uniformly in R_1 or R_2. Since N_1 and N_2 are arbitrary positive numbers and N_3 is an arbitrary positive number $< (2T^{m-1}m^m)^{-1}$ with $T > \tau$, the stated result follows.

4. POLYNOMIAL SETS WHEN $n > 2$

Using series methods for ordinary differential equations, the solution of (1.1) for $n \geq 2$ can be expressed formally by the relation

$$u(x,t) = \sum_{j=0}^{n-1} \frac{t^j}{j!} \cdot O\left(\frac{\pm t^n D^m}{n^n}; \frac{j+1}{n}, \frac{j+2}{n}, \ldots, \frac{j+n}{n} \right) \phi_j(x). \tag{4.1}$$

When $j < n - 1$, the solution operators in this sum have at least one index that is less than one. For these series solution operators, we can write

$$\begin{aligned} O&\left(\pm \frac{t^n D^m}{n^n}; \frac{j+1}{n}, \ldots, \frac{j+n}{n} \right) \\ &= 1 + \frac{(\pm 1)t^n D^m}{(j+1)_n} O\left(\pm \frac{t^n D^m}{n^n}; \frac{j+1+n}{n}, \ldots, \frac{j+2n}{n} \right). \end{aligned} \tag{4.2}$$

The operator in the right member of this now fits the form required in Section 2, part C.

We now introduce the symbolism $P_k^{m,n,j,\pm}(x) = O(\pm t^n D^m/n^n; (j+1)/n, \dots, (j+n)/n)x^k$ for the polynomial solutions of (1.1) corresponding to the values $j = 0, 1, \dots, n-1$. To obtain these, we write the corresponding generating function, namely

$$G^{m,n,j}(x,t,a) = O\left(\pm\frac{t^n D^m}{n^n}; \frac{j+1}{n}, \dots, \frac{j+n}{n}\right)e^{ax}$$
$$= O\left(\pm\frac{t^n a^m}{n^n}; \frac{j+1}{n}, \dots, \frac{j+n}{n}\right)e^{ax}. \qquad (4.3)$$

Upon multiplying the series expansions of the two functions appearing in the last member of this, we get

$$G^{m,n,j}(x,t,a)$$
$$= \left(\sum_{r=0}^{\infty}\frac{(\pm1)^r (t/n)^{nr} a^{mr}}{((j+1)/n)_r((j+2)/n)_r \cdots ((j+n)/n)_r}\right)\cdot\left(\sum_{l=0}^{\infty}\frac{a^l x^l}{l!}\right)$$
$$= \sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\pm1)^r (t/n)^{nr} x^l a^{mr+l}}{((j+1)/n)_r((j+2)/n)_r \cdots ((j+n)/n)_r l!}$$
$$= \sum_{k=0}^{\infty}\frac{a^k}{k!}\left(\sum_{r=0}^{[k/m]}\frac{k!(\pm1)^r t^{nr} x^{k-mr}}{n^{nr}((j+1)/n)_r \cdots ((r+n)/n)_r (k-mr)!}\right), \qquad (4.4)$$

where we have made the change of indices $k = mr + l$ to obtain the last sum. From this, we obtain the explicit polynomial formulas, namely

$$P_k^{m,n,j,\pm}(x,t) = \sum_{r=0}^{[k/m]}\frac{(\pm1)^r k! t^{nr} x^{k-mr}}{n^{nr}((j+1)/n)_r \cdots ((r+n)/n)_r (k-mr)!},$$
$$j = 0, 1, \dots, n-1. \qquad (4.5)$$

Applying the right hand member of (4.2) to x^k, we also find that $P_k^{m,n,j,\pm}(x,t) = x^k$ if $m > k$ and

$$P_k^{m,n,j,\pm}(x,t)$$
$$= x^k \pm \frac{k!\cdot t^n}{(j+1)_n (k-m)!}O\left(\frac{\pm t^n D^m}{n^n}; \frac{j+1+n}{n}, \dots, \frac{j+2n}{n}\right)x^{k-m} \qquad (4.6)$$

if $k \geq m$ for $j = 0, 1, \ldots, n-2$. This last relation will be employed in Sections 5 and 6 to determine bounds on the polynomials $P_k^{m,n,j,\pm}(x,t)$.

Next, we derive x derivative formulas for these polynomials. These follow immediately by differentiating the formula defining the $P_k^{m,n,j,\pm}(x,t)$. Thus we find

$$\begin{aligned} D^r P_k^{m,n,j,\pm}(x,t) &= D^r O\left(\pm\frac{t^n D^m}{n^n}; \frac{j+1}{n}, \ldots, \frac{j+n}{n}\right) x^k \\ &= O\left(\pm\frac{t^n D^m}{n^n}; \frac{j+1}{n}, \ldots, \frac{j+n}{n}\right)(D^r x^k) \\ &= \frac{k!}{(k-r)!} O\left(\pm\frac{t^n D^m}{n^n}; \frac{j+1}{n}, \ldots, \frac{j+n}{n}\right) x^{k-r} \\ &= \frac{k!}{(k-r)!} P_{k-r}^{m,n,j,\pm}(x,t) \end{aligned} \tag{4.7}$$

if $k \geq r$ and 0 otherwise.

Finally, it is left to the reader to use (4.5) and some factoring to verify the following time derivative relations:

$$\begin{aligned} \text{(a)}\quad & \frac{\partial}{\partial t} P_k^{m,n,j,\pm}(x,t) \\ &= \frac{j}{t}\left(P_k^{m,n,j-1,\pm}(x,t) - P_k^{m,n,j,\pm}(x,t)\right), \qquad j = 1, 2, \ldots, n-1 \\ \text{(b)}\quad & \frac{\partial}{\partial t} P_k^{m,n,0,\pm}(x,t) \\ &= \begin{cases} 0 & \text{if } k < m \\ \pm\dfrac{k!\,t^{n-1}}{(n-1)!(k-m)!} P_k^{m,n,n-1,\pm}(x,t) & \text{if } k \geq m. \end{cases} \end{aligned} \tag{4.8}$$

Higher order time derivatives of the $P_k^{m,n,j,\pm}(x,t)$ can be obtained by repeatedly applying these results.

5. POLYNOMIAL EXPANSIONS WHEN $m \leq n$

We now consider the problem of writing expansions of solutions of the problem (1.1) in terms of the polynomial sets $P_k^{m,n,j,\pm}(x,t)$ when $n \geq m$

and then determine regions of convergence of these expansions. In view of the formula (4.2), it suffices to consider two typical cases: (i) $(\partial^j/\partial t^j)u(x,t)|_{t=0} = \phi(x)$ for a fixed $j \le n-2$ with $(\partial^p/\partial t^p)u(x,t)|_{t=0} = 0$ for $p \in \{0,1,\dots,n-1\} \setminus \{j\}$ and (ii) $(\partial^{n-1}/\partial t^{n-1})u(x,t)|_{t=0} = \phi(x)$ and $(\partial^p/\partial t^p)u(x,t)|_{t=0} = 0$ for $p \in \{0,1,\dots,n-2\}$. For an example of case (i), we make the selection $j=0$. The details involved for this case are essentially the same as the ones needed for the other values of j. When $n > m \ge 1$, there is some flexibility in obtaining regions of convergence of expansions of solutions corresponding to analytic $\phi(x)$. The utility of this will be exhibited in Theorem 5.2 below.

Case (i). Let $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in a disk centered at the origin with radius R (hence, $\phi(x)$ is R-analytic for $|x| < R$). We wish to examine where the sum

$$u(x,t) = \sum_{k=0}^{\infty} a_k P_k^{m,n,0,\pm}(x,t) \tag{5.1}$$

converges and determine if it defines a solution of (1.1) corresponding to the above assigned data conditions. To find where it converges, we must first obtain bounds on the polynomials $P_k^{m,n,0,\pm}(x,t)$. To do this, we first apply both sides of (4.2), with $j=0$, to x^k. If $k<m$, this yields $P_k^{m,n,0,\pm}(x,t) = x^k$. However, if $k \ge m$, this application leads to the relation

$$P_k^{m,n,0,\pm}(x,t) = x^k \pm \frac{k!t^n}{(k-m)!(n-1)!}\left\{O\left(\pm\frac{t^nD^m}{n^n};\nu_1,\dots,\nu_n\right)x^{k-m}\right\}, \tag{5.2}$$

where $\nu_l = (1+l)/n$. Before bounding this, we write a multiple integral formula for the expression in brackets by decomposing the O operator into g_a type operators and then write these in terms of integrals of exponential operators. There are a variety of ways of carrying out this decomposition depending upon how we switch around the factors of $\pm(t^nD^m)/n^n$ when using the qip.

Let λ denote a positive number with $n - m\lambda \ge 0$, let c be a positive number, and assume, for the moment, that $t \ge 0$ (we discuss the possible choices for λ and c when we consider regions of convergence of the expansions). Applying the reduction (2.8) m successive times to the operator O in (5.2), we can write

$$
\begin{aligned}
O&\left(\pm\frac{t^nD^m}{n^n};\nu_1,\dots,\nu_n\right)x^{k-m}\\
&=\frac{1}{(2\pi)^m}\int_{[0,2\pi]}O\left(\pm\frac{t^{n-m\lambda}}{n^nc^m};\nu_{m+1},\dots,\nu_n\right)\\
&\quad\times\left\{g_{\nu_1}(ct^\lambda e^{i\theta_1}D)\prod_{l=2}^m g_{\nu_l}(ct^\lambda e^{i(\theta_l-\theta_{l-1})}D)\right\}x^{k-m}\,d\Theta_m. \qquad (5.3)
\end{aligned}
$$

If $n=m$, we choose $c=1/n$ and $\lambda=1$ in this. In this case, the O function appearing in the integral in (5.3) reduces to 1. Next, apply the formula (2.7) to each of the g_a operators in this to obtain translation operators and, hence, the integral formula required, namely

$$
\begin{aligned}
O&\left(\pm\frac{t^nD^m}{n^n};\nu_1,\dots,\nu_n\right)x^{k-m}\\
&=\frac{(\nu_1-1)\cdots(\nu_m-1)}{(2\pi)^m}\int_{[0,1]}\int_{[0,2\pi]}\left(\prod_{l=1}^m(1-\sigma_l)^{\nu_l-2}\right)\\
&\quad\times O\left(\pm\frac{t^{n-m\lambda}}{n^nc^m}e^{-i\theta_m};\nu_{m+1},\dots,\nu_n\right)\\
&\quad\times\left[e^{t^\lambda c[\sigma_1e^{i\theta_1}+\sigma_2e^{i(\theta_2-\theta_1)}+\cdots+\sigma_me^{i(\theta_m-\theta_{m-1})}]D}x^{k-m}\right]d\Theta_m\,d\Sigma_m\\
&=\frac{(\nu_1-1)\cdots(\nu_m-1)}{(2\pi)^m}\int_{[0,1]}\int_{[0,2\pi]}\left(\prod_{l=1}^m(1-\sigma_l)^{\nu_l-2}\right)\\
&\quad\times O\left(\pm\frac{t^{n-m\lambda}}{n^nc^m}e^{-i\theta_m};\nu_{m+1},\dots,\nu_n\right)\\
&\quad\times\Big(x+t^\lambda c\big[\sigma_1e^{i\theta_1}+\sigma_2e^{i(\theta_2-\theta_1)}\\
&\qquad\qquad+\cdots+\sigma_me^{i(\theta_m-\theta_{m-1})}\big]\Big)^{k-m}d\Theta_m\,d\Sigma_m. \qquad (5.4)
\end{aligned}
$$

Since $0\le\sigma_l\le1$ for $l=1,2,\dots,m$, it follows that

$$
\begin{aligned}
&\left|x+t^\lambda c\big[\sigma_1e^{i\theta_1}+\sigma_2e^{i(\theta_2-\theta_1)}+\cdots\ \sigma_me^{i(\theta_m-\theta_{m-1})}\big]\right|^{k-m}|\\
&\qquad\le\left(|x|+mc|t|^\lambda\right)^{k-m}. \qquad (5.5)
\end{aligned}
$$

Upon using this to compute the absolute value of the last member of (5.4), we get

$$\begin{aligned}
&\left| O\left(\pm \frac{t^n D^m}{n^n}; \nu_1, \dots, \nu_n \right) x^{k-m} \right| \\
&\qquad \le (\nu_1 - 1) \cdots (\nu_m - 1) \int_{[0,1]} O\left(\frac{t^{n-m\lambda}}{n^n c^m}; \nu_{m+1}, \dots, \nu_n \right) \\
&\qquad\qquad \times \left(\prod_{l=1}^{m} (1 - \sigma_l)^{\nu_l - 2} \right) \left(|x| + mc|t|^\lambda \right)^{k-m} d\Sigma_m \\
&\qquad = O\left(\frac{t^{n-m\lambda}}{n^n c^m}; \nu_{m+1}, \dots, \nu_n \right) \left(|x| + mc|t|^\lambda \right)^{k-m}.
\end{aligned} \tag{5.6}$$

We now take the absolute value of the right member (5.2), by employing this last inequality, to obtain the following bound on $|P_k^{m,n,0,\pm}(x,t)|$:

$$\begin{aligned}
&\left| P_k^{m,n,0,\pm}(x,t) \right| \\
&\quad \le |x|^k + \frac{k!|t|^n}{(k-m)!(n-1)!} O\left(\frac{|t|^{n-m\lambda}}{n^n c^m}; \nu_1, \dots, \nu_n \right) \left(|x| + mc|t|^\lambda \right)^{k-m}.
\end{aligned} \tag{5.7}$$

Finally, suppose that $t < 0$ in the first member of (5.3). We can replace t by $-(-t)$ in that operator O and then use the same argument as above to deduce (5.7).

Now, consider the expansion (5.1) when $m = n$. Taking its absolute value and using (5.7), it follows that

$$|u(x,t)| \le \sum_{k=0}^{\infty} |a_k| \, |x|^k + \frac{|t|^n}{(n-1)!} \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} |a_k| (|x| + |t|)^{k-m}.$$

Applying the root test, it follows that the right hand member of this converges if $|x| + |t| < R$ (i.e., an open square). On the other hand, if $1 \le m < n$, it follows by similar reasoning that the series in (5.1) converges if $|x| + mc|t|^\lambda < R$ since the O function appearing in the right member of (5.7) is independent of k. We see from this that there are numerous possible choices for λ and c that define somewhat different convergence regions. Two of these regions are defined by $|x| + (m/n)|t| < R$ (by taking $c = 1/n$ and $\lambda = 1$) and $|x| + (m/n^{n/m})|t|^{n/m} < R$ (with $c = 1/n^{n/m}$

and $\lambda = n/m$). We call these R-regions. Suppose we replace R by R^* in these inequalities where $0 < R^* < R$ and form the closures of these R^* regions. We denote these by $C(R^*)$. Then we see that the series (5.1) converges uniformly in $C(R^*)$. The formula (4.7) and the construction of bounds on these similar to (5.7) can be used repeatedly to show that the x derivatives of the series defining $u(x,t)$ converge uniformly in $C(R^*)$ and hence converge in the associated R-regions. In a similar way, the formula (4.8)(b) can be used to show that the first t derivative of the series defining $u(x,t)$ converges uniformly in $C(R^*)$. Further, since the formula (4.8)(b) contains the factor t^{n-1}, successive applications of the product rule of differentiation and (4.8)(a) permit us to show that the jth t-derivative of the series defining $u(x,t)$ converges uniformly in $C(R^*)$ for $j = 2, 3, \dots, n - 1$. Since the $P_k^{m,n,0,\pm}(x,t)$ satisfy the equation in (1.1), it follows that the series defining $u(x,t)$ for $n \geq m$ defines a solution of the partial differential equation in (1.1) in all of the $C(R^*)$ and hence in the R-regions. Finally,

$$u(x,0) = \sum_{k=0}^{\infty} a_k P_k^{m,n,0,\pm}(x,0) = \sum_{k=0}^{\infty} a_k x^k$$

for $|x| < R$. The formulas in (4.8) can be used to show that the derivatives with respect to t of orders up to $n - 1$ vanish in the R-regions. With this, we have proved

THEOREM 5.1. *Let $\phi(x)$ be R-analytic for $|x| < R$. If $n = m$, the series* (5.1) *converges to a solution of the problem* (1.1) *corresponding to the initial conditions $u(x,0) = \phi(x)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$, $j = 1, 2, \dots, n - 1$ in the region $|x| + |t| < R$. If $n > m$, the series* (5.1) *corresponding to these same initial conditions converges to a solution of the problem* (1.1) *in the region $|x| + mc|t|^\lambda < R$ where c is a positive constant and λ is a positive constant that satisfies the condition $n - \lambda m \geq 0$.*

Let us note that if we select $\lambda = n/m$ when $n > m$, then the O function in the right member of (5.7) reduces to a constant and a convergence region can be given by the inequality $|x| + mc|t|^{n/m} < R$. Such a region is symmetric about the x and the t axes. Now let (α, β) be an arbitrary point in the positive quadrant of the (x,t) plane with $0 \leq \alpha < R$. We wish to show that this point and its reflections through the x and t axes can be contained within a region of convergence. Choose R^* such that $\alpha < R^* < R$. Let $\beta^* > \beta$ and fit the curve $|x| + mc|t|^{n/m} = R^*$ through the point (α, β^*). This requires that $c = (R^* - \alpha)/(m(\beta^*)^{n/m})$ which is a positive constant. From this, we see that the point (α, β) lies within the compact set $|x| + (R^* - \alpha)/((\beta^*)^{n/m})|t|^{n/m} \leq R^*$ where the series converges uniformly. From this argument, we have

THEOREM 5.2. *Let $\phi(x)$ be R-analytic for $|x| < R$. If $n > m \geq 1$, the series* (5.1) *converges to a solution of the problem* (1.1) *corresponding to the initial conditions $u(x, 0) = \phi(x)$, $\partial^j u(x, t)/\partial t^j|_{t=0} = 0$, $j = 1, 2, \ldots, n-1$ at any point in the strip $|x| < R$ of the $x - t$ plane.*

Remark. When $n = m = 2$, this region of convergence agrees with the one obtained by D. V. Widder for the case of the wave equation (plus sign in (1.1)). However, for the case of (1.1) involving the minus sign (the Laplace equation), Widder obtained the disk of convergence $x^2 + t^2 < R^2$. This discrepancy with Theorem 5.1 follows from the way we computed the absolute value in (5.5) for the general case $n \geq m$. Had we split the expression under the absolute value signs in the first member of this into real and imaginary parts, then the usual modulus would have yielded a complicated expression in the variables θ_j, $j = 1, 2, \ldots, n$. This, in turn, would then have made it difficult to obtain convenient bounds for the polynomials $P_k^{m,n,0,\pm}(x, t)$. In the Laplace case, a simpler expression involving only one θ variable appears in the expression between the absolute value signs in the first member of the inequality (5.5). From this, one can conveniently use the modulus to obtain the disk of convergence (see [4]).

Case (ii). For this, we replace the sum in (5.1) by

$$u(x, t) = \sum_{k=0}^{\infty} a_k t^{n-1} P_k^{m,n,n-1,\pm}(x, t)/(n-1)! \tag{5.8}$$

with the $P_k^{m,n,n-1,\pm}(x, t)$ defined operationally by the formula

$$P_k^{m,n,n-1,\pm}(x, t) = O\left(\pm \frac{t^n D^m}{n^n}; \frac{n}{n}, \frac{n+1}{n}, \ldots, \frac{2n-1}{n}\right) x^k. \tag{5.9}$$

By taking $\nu_r = (n - 1 + r)/n$ for $r = 1, 2, \ldots, n$, we can replace the right hand side of this by a multiple integral (by using the g_a reduction as in deriving (5.4)) and then obtain a bound on this integral. This bound is given by

$$\left|P_k^{m,n,n-1,\pm}(x, t)\right| \leq O\left(\frac{t^{n-\lambda m}}{n^n c^m}; \nu_{m+1}, \ldots, \nu_n\right)\left(|x| + mc|t|^{\lambda}\right)^k, \tag{5.10}$$

where λ and c are selected as in case (i) above. With this bound, we obtain, with a slight exception, the same regions of convergence as in case (i): $|x| + |t| < R$ when $n = m$ and $|x| + mc|t|\lambda < R$ when $n > m$. The exception occurs when $t = 0$ since the series (5.8) then converges for all real x. From the fact that

$$D^r P_k^{m,n,n-1,\pm}(x,t) = \frac{k!}{(k-r)!} O\left(\pm \frac{t^n D^m}{n^n}; \frac{n}{n}; \frac{n+1}{n}; \cdots; \frac{2n-1}{n}\right) x^{k-r}$$

if $k \geq r$ and 0 otherwise, we can show that the derivatives of the series (5.8) through order $n-1$ with respect to x converge uniformly in the $C(R^*)$ subsets of the regions of convergence. The computation of the t derivatives of the series (5.8) can be carried out by using (4.8). Because of the factor $t^{n-1}/(n-1)!$ in each term in the series (5.8), it is readily checked that the initial conditions required in case (ii) are satisfied. We have

THEOREM 5.3. *Let $\phi(x)$ be R-analytic for $|x| < R$. If $n = m$, the series* (5.8) *converges to a solution of the problem* (1.1) *corresponding to the initial conditions $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$, $j = 0, 1, \ldots, n-2$, and $\partial^{n-1} u(x,t)/\partial t^{n-1}|_{t=0} = \phi(x)$ in the region $\{|x| + |t| < R\} \cup \{t = 0\}$. If $n > m$, the series* (5.1) *corresponding to these initial conditions converges to a solution of problem* (1.1) *in the region $\{|x| + mc|t|^{\lambda} < R\} \cup \{t = 0\}$ where c is a positive constant and λ is a positive constant that satisfies the condition $n - \lambda m \geq 0$.*

It is left to the reader to extend Theorem 5.2 to this case.

6. POLYNOMIAL EXPANSIONS WHEN $m > n$

In view of our discussions in Section 5, it suffices to consider the expansion of a solution of (1.1) subject to the initial conditions $u(x,0) = \phi(x)$, $\partial^j u(x,0)/\partial t^j = 0$ for $j = 1, 2, \ldots, n-1$ where $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$ is entire of appropriate growth. By (4.2), the solution of this problem can be expressed in the symbolic operator form

$$u(x,t) = \phi(x) \pm \frac{t^n D^m}{n!} O\left(\pm \frac{t^n D^m}{n^n}; \nu_1, \nu_2, \ldots, \nu_n\right) \phi(x), \quad (6.1)$$

where $\nu_r = (r+n)/n$ for $r = 1, 2, \ldots, n$. If we replace $\phi(x)$ in this by x^k, we get

$$P_k^{m,n,0,\pm}(x,t) = \begin{cases} x^k & \text{if } m > k \\ x^k \pm \dfrac{k!\,t^n}{n!(k-m)!} O\left(\pm \dfrac{t^n D^m}{n^n}; \nu_1, \ldots, \nu_n\right) x^{k-m} & \text{if } m \leq k. \end{cases} \quad (6.2)$$

Case (i). *n divides m.* Let $s = m/n$. We decompose the operator O in the second member of the right hand side of (6.2) into g_a type operators. Using the switching property and the decomposition approach of Section 5, we can write

$$O\left(\pm\frac{t^n D^m}{n^n};\nu_1,\ldots,\nu_n\right)x^{k-m}$$
$$=\frac{1}{(2\pi)^{n-1}}\int_{[0,2\pi]}\left\{g_{\nu_1}\left(\pm\frac{te^{i\theta_1}D^s}{n}\right)g_{\nu_2}\left(\frac{te^{i(\theta_2-\theta_1)}D^s}{n}\right)\right.$$
$$\left.\times\cdots g_{\nu_n}\left(\frac{te^{-i\theta_{n-1}}D^s}{n}\right)x^{k-m}\right\}d\Theta_{n-1}.$$

Upon applying the integral formulas (2.7) for these g_{ν_j}, we can rewrite this in the form

$$O\left(\pm\frac{t^n D^m}{n^n};\nu_1,\ldots,\nu_n\right)x^{k-m}$$
$$=\frac{(\nu_1-1)(\nu_2-1)\cdots(\nu_n-1)}{(2\pi)^{n-1}}$$
$$\times\int_{[0,1]}\int_{[0,2\pi]}(1-\sigma_1)^{\nu_1-2}\cdots(1-\sigma_n)^{\nu_n-2}$$
$$\times\left\{e^{(t/n)(\pm\sigma_1 e^{i\theta_1}+\sigma_2 e^{i(\theta_2-\theta_1)}+\sigma_3 e^{i(\theta_3-\theta_2)}+\cdots+\sigma_n e^{-i\theta_{n-1}})D^s}x^{k-m}\right\}d\Theta_{n-1}\,d\Sigma_n$$
$$=\frac{(\nu_1-1)(\nu_2-1)\cdots(\nu_n-1)}{(2\pi)^{n-1}}$$
$$\times\int_{[0,1]}\int_{[0,2\pi]}(1-\sigma_1)^{\nu_1-2}\cdots(1-\sigma_n)^{\nu_n-2}$$
$$\times H^{s,+}_{k-m}\left(x,\frac{t}{n}\left(\pm\sigma_1 e^{i\theta_1}+\sigma_2 e^{i(\theta_2-\theta_1)}+\cdots+\sigma_n e^{-i\theta_{n-1}}\right)\right)$$
$$\times d\Theta_{n-1}\,d\Sigma_n, \tag{6.3}$$

where $H^{s,+}_{k-m}(x,t)$ is a generalized heat polynomial which was defined in Section 3. Since $|(t/n)(\sigma_1 e^{i\theta_1}+\sigma_2 e^{i(\theta_2-\theta_1)}+\cdots+\sigma_n e^{-i\theta_{n-1}})|\le|t|$, we can apply (3.10) to obtain

$$\left| H_{k-m}^{s,+}\left(x, \frac{t}{n}\left(\pm\sigma_1 e^{i\theta_1} + \sigma_2 e^{i(\theta_2-\theta_1)} + \cdots + \sigma_n e^{-i\theta_{n-1}} \right)\right)\right| \\ \leq \left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f(t), \tag{6.4}$$

where $f(t) = \int_{[0,\infty)} e^{-\sum_{j=1}^{s-1}\zeta_j} e^{2^{\rho-1}Ts^\rho|t|^{\rho/s}(\zeta_1+\zeta_2+\cdots+\zeta_{s-1})^{\rho(s-1)/s}}\, dZ_{s-1}$. We observe that if $\rho < s/(s-1)$, then $f(t)$ is defined for all t and if $\rho = s/(s-1)$, then $f(t)$ is defined if $|t| < (2T^{s-1}s^s)^{-1}$. We can now use (6.4) with (6.3) to obtain the bound

$$\left| O\left(\pm\frac{t^n D^m}{n^n}; \nu_1, \ldots, \nu_n\right) x^{k-m}\right| \\ \leq \left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f(t). \tag{6.5}$$

Finally, we can use this in (6.2) to establish that

$$\left| P_k^{m,n,0,\pm}(x,t)\right| \\ \leq \begin{cases} |x|^k & \text{if } k < m \\ |x|^k + \dfrac{|t|^n k!}{(k-m)!n!}\left(\dfrac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f(t) \\ \quad \text{if } k \geq m. \end{cases} \tag{6.6}$$

Let us now assume that $\phi(x)$ has growth (ρ, τ) with $\rho \leq s/(s-1)$. Then we have $|a_k| \leq (e\rho/k)^{k/\rho}(\tau+\varepsilon)^{k/\rho}$ for $k \geq K$ where $\varepsilon > 0$. Let $u^*(x,t) = \sum_{k=0}^\infty a_k P_k^{m,n,0,\pm}(x,t)$. Upon taking the absolute value of this, we have

$$|u^*(x,t)| \\ \leq \sum_{k=0}^\infty |a_k| \left| P_k^{m,n,0,\pm}(x,t)\right| \\ \leq \left(\sum_{k=0}^K |a_k| \left| P_k^{m,n,0,\pm}(x,t)\right| \right) + \sum_{k=K+1}^\infty |a_k| \left(|x|^k + \frac{|t|^n k!}{n!(k-m)!} \right. \\ \left. \times \left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f(t)\right)$$

$$\leq \sum_{k=0}^{K} |a_k| \left| P_k^{m,n,0,\pm}(x,t) \right| + e^{2^{\rho-1}T|x|^\rho} |t|^n f(t)(n!)^{-1}$$
$$\times \left\{ \sum_{k=K+1}^{\infty} \frac{k!}{(k-m)!} \left(\frac{e\rho}{k} \right)^{k/\rho} (\tau + \varepsilon)^{k/\rho} \right.$$
$$\left. \times \left(\frac{k-m}{\rho T} \right)^{(k-m)/\rho} e^{-(k-m)/\rho} \right\}. \tag{6.7}$$

An application of the ratio test shows that the infinite sum in the final member of this converges if $\tau + \varepsilon < T$.

First, suppose that $\rho < s/(s-1)$ and select large positive numbers M and N and restrict x and t so that $|x| \leq M$ and $|t| \leq N$. Then the series for $u^*(x,t)$ converges uniformly in the rectangle $[-M, M] \times [-N, N]$. Using the formulas (4.7) and (4.10) and carrying out calculations similar to that in (6.7), we can show that the x derivatives of orders up to n and the t derivatives of orders up to m of the series defining $u^*(x,t)$ also converge uniformly in this rectangle. Since the $P_k^{m,n,0,\pm}(x,t)$ satisfy Eq. (1.1), it follows that $u^*(x,t)$ satisfies Eq. (1.1) in this rectangle. Again using (4.10), it is easily checked that the initial conditions are satisfied in this rectangle. But since M and N are arbitrary, this establishes

THEOREM 6.1. *Suppose that n divides m with $s = m/n$. Let $\phi(x)$ be entire of growth (ρ, τ) with $0 < \rho < s/(s-1)$. Then the series $u(x,t) = \sum_{k=0}^{\infty} a_k P_k^{m,n,0,\pm}(x,t)$ converges to a solution of Eq. (1.1) and satisfies the conditions $u(x,0) = \phi(x)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$ for $j = 1, 2, \ldots, n-1$ for all x and t.*

If $\rho = s/(s-1)$, we can modify the above argument by selecting positive N so that $N < (2T^{s-1}s^s)^{-1}$. Within this new rectangle $[-M, M] \times [-N, N]$, the same arguments as above apply. This leads to

THEOREM 6.2. *Suppose that n divides m with $s = m/n$. Let $\phi(x)$ be entire of growth (ρ, τ) with $\rho = s/(s-1)$. Then the series $u(x,t) = \sum_{k=0}^{\infty} a_k P_k^{m,n,0,\pm}(x,t)$ converges to a solution of Eq. (1.1) and satisfies the conditions $u(x,0) = \phi(x)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$ for $j = 1, 2, \ldots, n-1$ for all x and for $|t| < (2\tau^{s-1}s^s)^{-1}$.*

Case (ii). *n does not divide m.* We can write $m = ns + r$ where $1 \leq r \leq n-1$. We consider the same form of the initial value problem as in case (i). We will reduce the development here to the same method as in case (i) by obtaining a bound on the function $O(\pm(t^n D^m/n^n); \nu_1, \ldots, \nu_n)x^{k-m}$ which appears in (6.2). However, the decomposition of the operator O in (6.2) will need to be carried out somewhat differently in order to deduce optimal growth conditions on the function $\phi(x)$. As before, we write a g_{ν_j} decomposition of the operator O, and using the switching property, place the terms tD^{s+1}/n in r of the g_{ν_j}'s and the terms tD^s/n in the remaining g_{ν_j}'s. Thus, we can write

$$\begin{aligned}
&O\left(\pm\frac{t^nD^m}{n^n};\nu_1,\dots,\nu_n\right)x^{k-m}\\
&\quad=\frac{1}{(2\pi)^{n-1}}\int_{[0,2\pi]}\left\{\left[g_{\nu_1}\left(\pm\frac{t}{n}e^{i\theta_1}D^{s+1}\right)g_{\nu_2}\left(\frac{t}{n}e^{i(\theta_2-\theta_1)}D^{s+1}\right)\right.\right.\\
&\qquad\qquad\left.\times\cdots g_{\nu_r}\left(\frac{t}{n}e^{i(\theta_r-\theta_{r-1})}D^{s+1}\right)\right]\\
&\qquad\times\left[g_{\nu_{r+1}}\left(\frac{t}{n}e^{i(\theta_{r+1}-\theta_r)}D^s\right)g_{\nu_{r+2}}\left(\frac{t}{n}e^{i(\theta_{r+2}-\theta_{r+1})}D^s\right)\right.\\
&\qquad\qquad\left.\left.\times\cdots g_{\nu_n}\left(\frac{t}{n}e^{-i\theta_{n-1}}D^s\right)\right]\right\}x^{k-m}\,d\Theta_{n-1}.
\end{aligned}$$

Again, if we replace the g_{ν_j} operators by their integral forms, this becomes

$$\begin{aligned}
&O\left(\pm\frac{t^nD^m}{n^n};\nu_1,\dots,\nu_n\right)x^{k-m}\\
&\quad=\frac{(\nu_1-1)\cdots(\nu_n-1)}{(2\pi)^{n-1}}\int_{[0,1]}\int_{[0,2\pi]}(1-\sigma_1)^{\nu_1-2}\cdots(1-\sigma_n)^{\nu_n-2}\\
&\qquad\times W(x,t,\Sigma_n,\Theta_{n-1})\,d\Theta_{n-1}\,d\Sigma_n,
\end{aligned}\tag{6.8}$$

where

$$\begin{aligned}
&\text{(a)}\quad W(x,t,\Sigma_n,\Theta_{n-1})=e^{t(aD+b)D^s}x^{k-m}\\
&\text{(b)}\quad a=\frac{1}{n}\left(\pm\sigma_1e^{i\theta_1}+\sigma_2e^{i(\theta_2-\theta)_1}+\cdots+\sigma_re^{i(\theta_r-\theta_{r-1}}\right)\\
&\text{(c)}\quad b=\frac{1}{n}\left(\sigma_{r+1}e^{i(\theta_{r+1}-\theta_r)}+\cdots+\sigma_ne^{-i\theta_n}\right).
\end{aligned}\tag{6.9}$$

From the range of values on the σ_j we note that

$$|a|\le r/n<1\qquad\text{and}\qquad|b|\le(n-r)/n<1.\tag{6.10}$$

In order to deduce bounds on the polynomials $P_k^{m,n,0,\pm}(x,t)$, we must first obtain a bound on the function $W(x,t,\Sigma_n,\Theta_{n-1})$. To do this, we must first express the right member of (6.9) in multiple integral form. First, write $t = \pm|t|$ and then apply formula (2.5) of [3] with a replaced in that formula by $\pm|t|(aD+b)$ and b by D^s. We get

$$\begin{aligned}
&W(x,t,\Sigma_n,\Theta_{n-1})\\
&\quad = e^{\pm|t|(aD+b)D^s} x^{k-m}\\
&\quad = \int_0^\infty e^{-\zeta_1}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{\pm|t|e^{i\varphi_1}(aD+b)} e^{\zeta_1 e^{-i\varphi_1}D^s} x^{k-m}\right\} d\varphi_1\, d\zeta_1\\
&\quad = \int_0^\infty e^{-\zeta_1}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{\pm|t|^{1/(s+1)}\zeta_1^{s/(s+1)}e^{i\varphi_1}(aD+b)}\right.\\
&\qquad\qquad \left.\times\left[e^{|t|^{s/(s+1)}\zeta_1^{1/(s+1)}e^{-i\varphi_1}D^s}\right] x^{k-m}\right\} d\varphi_1\, d\zeta_1,
\end{aligned}$$

where the last member follows from the third member by using the switching property. We then use the formula (2.6) for the exponential operator in braces with $l = s$, $\alpha = |t|^{s/(s+1)}\zeta_1^{1/(s+1)}e^{-i\varphi_1}$, and integration variables $\varphi_2,\ldots,\varphi_s$ and $\zeta_2,\ldots,\zeta_s$ to obtain

$$\begin{aligned}
&W(x,t,\Sigma_n,\Theta_{n-1})\\
&\quad = \frac{1}{(2\pi)^s}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\Sigma_{j=1}^s\zeta_j} e^{\pm|t|^{1/(s+1)}\zeta_1^{s/(s+1)}b}\\
&\qquad \times e^{|t|^{1/(s+1)}[a\zeta_1^{1/(s+1)}e^{i\varphi_1} + \zeta_1^{1/(s+1)}\zeta_2^{(s-1)/(s+1)}e^{i(\varphi_2-\varphi_1)} + \cdots + \zeta_s^{s/(s+1)}e^{-i\varphi_s}]D}\\
&\qquad \times x^{k-m}\, d\Phi_s\, dZ_s\\
&\quad = \frac{1}{(2\pi)^s}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\Sigma_{j=1}^s\zeta_j} e^{\pm|t|^{1/(s+1)}\zeta_1^{s/(s+1)}b}\\
&\qquad \times\Big(x + |t|^{1/(s+1)}\Big[\pm a\zeta_1^{s/(s+1)}e^{i\varphi_1} + \zeta_1^{1/(s+1)}\zeta_2^{(s-1)/(s+1)}e^{i(\varphi_2-\varphi_1)}\\
&\qquad\qquad + \cdots + \zeta_s^{s/(s+1)}e^{-i\varphi_s}\Big]\Big)^{k-m} d\Phi_s\, dZ_s,
\end{aligned} \tag{6.11}$$

where $d\Phi_s = d\varphi_1 d\varphi_2 \cdots d\varphi_s$. Taking the absolute value of this and using (6.10), we find

$$\begin{aligned}
&\left|W(x,t,\Sigma_n,\Theta_{n-1})\right|\\
&\quad \le \frac{1}{(2\pi)^s}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\Sigma_{j=1}^s\zeta_j} e^{|t|^{1/(s+1)}\zeta_1^{s/(s+1)}}\\
&\qquad \times\Big|x + |t|^{1/(s+1)}\Big[\pm a\zeta_1^{s/(s+1)}e^{i\varphi_1} + \zeta_1^{1/(s+1)}\zeta_2^{(s-1)/(s+1)}e^{i(\varphi_2-\varphi_1)}\\
&\qquad\qquad + \cdots + \zeta_s^{s/(s+1)}e^{-i\varphi_s}\Big]\Big|^{k-m} d\Phi_s\, dZ_s
\end{aligned}$$

$$\leq \frac{1}{(2\pi)^s}\int_{[0,\infty)}\int_{[0,2\pi]} e^{-\sum_{j=1}^s \zeta_j} e^{|t|^{1/(s+1)}\zeta_1^{s/(s+1)}}$$
$$\times\Big[|x| + |t|^{1/(s+1)}(s+1)(\zeta_1+\cdots+\zeta_s)^{s/(s+1)}\Big]^{k-m}\, d\Phi_s\, dZ_s. \tag{6.12}$$

By (3.6)(b), the last member of this is bounded by the integral

$$\left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho}$$
$$\times\int_{[0,\infty)} e^{-\sum_{j=1}^s \zeta_j} e^{|t|^{1/(s+1)}\zeta_1^{s/(s+1)}} e^{T||x|+|t|^{1/(s+1)}(s+1)(\zeta_1+\cdots+\zeta_s)^{s/(s+1)}|^\rho}\, dZ_s.$$

Now applying (3.6)(a), we can finally show that

$$\left|W(x,t,\Sigma_n,\Theta_{n-1})\right| \leq \left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f_1(t), \tag{6.13}$$

where

$$f_1(t) = \int_{[0,\infty)} e^{-\sum_{j=1}^s \zeta_j} e^{|t|^{1/(s+1)}\zeta_1^{s/(s+1)}} e^{2^{\rho-1}T|t|^{\rho/(s+1)}(s+1)^\rho(\zeta_1+\cdots+\zeta_s)^{\rho s/(s+1)}}\, dZ_s. \tag{6.14}$$

We observe that if $\rho < (s+1)/s$, then $f_1(t)$ is defined for all t. However, if $\rho = (s+1)/s$, then $f_1(t)$ is defined only if $|t| < (2T^s(s+1)^{s+1})^{-1}$. We can now use (6.13) along with (6.8) to finally establish the bound

$$\left|O\left(\pm\frac{t^nD^m}{n^n}; \nu_1,\ldots,\nu_n\right)x^{k-m}\right|$$
$$\leq \left(\frac{k-m}{\rho T}\right)^{(k-m)/\rho} e^{-(k-m)/\rho} e^{2^{\rho-1}T|x|^\rho} f_1(t). \tag{6.15}$$

With this, we can now establish bounds on the polynomials $P_k^{m,n,0,\pm}(x,t)$ from (6.2) just as we did in case (i) above. By then following the steps of case (i), we can finally prove

THEOREM 6.3. *Suppose that $m = ns + r$ with $1 \leq r \leq n-1$. Let $\phi(x)$ be entire of growth (ρ,τ) with $0 < \rho < (s+1)/s$. Then the series $u(x,t) = \sum_{k=0}^\infty a_k P_k^{m,n,0,\pm}(x,t)$ converges to a solution of Eq.* (1.1) *and satisfies the conditions $u(x,0) = \phi(x)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$ for $j = 1,2,\ldots,n-1$ for all x and t.*

THEOREM 6.4. *Suppose that* $m = ns + r$ *with* $1 \leq r \leq n - 1$. *Let* $\phi(x)$ *be entire of growth* (ρ, τ) *with* $\rho = (s+1)/s$. *Then the series* $u(x,t) = \sum_{k=0}^{\infty} a_k P_k^{m,n,0,\pm}(x,t)$ *converges to a solution of Eq.* (1.1) *and satisfies the conditions* $u(x,0) = \phi(x)$, $\partial^j u(x,t)/\partial t^j|_{t=0} = 0$ *for* $j = 1, 2, \ldots, n-1$ *for all* x *and for* $|t| < (2\tau^s(s+1)^{s+1})^{-1}$.

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